# On effective F-theory action in type IIA compactifications

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**Abstract**

Diaconescu, Moore and Witten proved that the partition function of type IIA string theory coincides (to the extent checked) with the partition function of M-theory. One of us (Kriz) and Sati proposed in a previous paper a refinement of the IIA partition function using elliptic cohomology and conjectured that it coincides with a partition function coming from F-theory. In this paper, we define the geometric term of the F-theoretical effective action on type IIA compactifications.

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In the special case when the first Pontrjagin class of spacetime vanishes, we also prove a version of the Kriz-Sati conjecture by extending the arguments of Diaconescu-Moore-Witten. We also briefly discuss why even this special case allows interesting examples.

# 1 Introduction

The purpose of this paper is to carry out the first stage of a program proposed in [15] of relating F-theory to type II partition functions. In the groundbreaking paper [2], Diaconescu, Moore and Witten established a solid connection between type IIA string theory and M-theory by comparing their partition functions. The subtlety lies mainly in the phase, which is determined by topological terms. In M-theory, we have the Chern-Simons term, in type IIA we have flux quantization using $K$-theory field strength. These definitions are quite different, and yet the partition functions coincide in the range examined, providing concrete computational evidence for string duality between type IIA and M-theory.

Yet [2] also contains a puzzle: in type IIB, when in the presence of the $H_3$ NS-NS field, the $K$-theoretical field strength is given by twisted $K$-theory, and ends up being incompatible with S-duality, in which $F_3$ and $H_3$ form a doublet. This puzzle was examined in [14] where it was determined that it cannot be solved by merely modifying the definition of twisted $K$-theory: one needs to quantize using a genuinely new generalized cohomology theory. A candidate emerged essentially simultaneously in the form of elliptic cohomology: in [13], the first author and Sati noticed that the quadratic structure of the $K$-theory IIA partition function has an analogue in elliptic cohomology. A similar construction works for IIB also, and has the intriguing aspect that, at least when we use the theory $TMF$ of topological modular forms, which can be regarded as a 'universal form of elliptic cohomology' (although it is not really elliptic cohomology in the proper sense, see comments in Section 6 below), turning on the $H_3$-field does not result in a twisting, but merely in multiplication by a cohomology class, which makes it possible that $H_3$ and $F_3$ could form a doublet in the same theory.

In [15], another connection was proposed, with a 12-dimensional theory known as F-theory. This theory was first considered by Vafa [20]. It was proposed that F-theory compactified on an elliptic curve $E$ is type IIB string theory, and that S-duality results in modular transformations on the space of the elliptic curves $E$. An action for F-theory relevant to that context was

proposed in [3]. Further dualities of F-theory compactified on an elliptic curve with type I and heterotic string theory were considered in [19, 4].

Based on this evidence, [15] proposed essentially the following program: The elliptic cohomology refinement of the quadratic structure introduced in [2] leads to a refinement of the $K$-theoretical field strength in type IIB string theory, and its partition function, which coincides with (the relevant part of) the partition function of F-theory, which would confirm the conjectured duality between type IIB and F-theory, simultaneously solving the S-theory puzzle.

Yet, this proposal has severe tests to withstand: By T-duality, there must be some analogous story for type IIA. In fact, there we are better equipped to study the partition functions, because of the work done in [2]. However, the problem is that one needs a universal definition of F-theory which would not be restricted to an elliptic curve fibration on IIB. In particular, we need to explain how the $G_5$ field, which seems intrinsic in the action of [3, 15] arises in a version of F-theory which contains type IIA string theory.

It is this part of the proposal which is investigated in the present paper. In other words, one of the main results of this paper is establishing a connection of F-theory with type IIA. Starting with an 11-dimensional M-theory spacetime $Y$, and its 12-dimensional spin cobordism $Z$, as in [2], we investigate an analogue of Witten's index formula [24] for the corrected Chern-Simons term on the *loop space* $LZ$, i.e. the space of maps from $S^1$ to $Z$. We use Witten's theory of the Dirac operator on loop space [22] to propose an F-theoretical analog of the Chern-Simons term. This is no longer a topological term (at least not in general), but it is a geometrical term in the sense that it depends only on the metric on the world volume. We propose a concrete formula for the geometric action term of F-theory compactified on type IIA spacetime. Further, we interpret the field strength $G_4$ on the loop space $LZ$ as an object encoding simulatneously the field strengths $G_4$ and $G_5$ on $Z$, thus establishing the connection with F-theory containing type IIB.

We do in fact investigate the compatibility of a fundamental F-theory with the basic T-duality between type IIA and IIB string theories. Thinking of "IIA" F-theory spacetime as a manifold with boundary which is the spacetime of M-theory on $X^9 \times S^1 \times S^1$, then we describe a process in which the first copy of $S^1$ shrinks to a point, while preserving the bulk. In a suitable case, this process gives F-theory fibered on IIB, which is one generalization

of T-duality. We also predict that the resulting theory without boundary should have a self-T-duality, which would exchange $G_4$ and $G_5$.

We then define a certain version of the elliptic partition function in type IIA proposed in [13], and indeed show that this partition function coincides with the partition function of the version of F-theory just discussed in the special case when the first Pontrjagin class vanishes. In the case of this rather special assumption we reduce the discussion to an easy extension of the methods of Diaconescu-Moore-Witten [2]. We also carry out a brief discussion showing that even this restrictive case includes interesting examples.

Our computation is restricted to certain values of the elliptic moduli parameter at which the phase factor in question is intrinsically defined in IIA and does not depend on the choice of $Z^{12}$.

The present paper is organized as follows: In section 2, we review Witten's formula for the Chern-Simons term in M-theory, and carry out a direct computation of the index involved. The direct index calculation turns out to be quite helpful in understanding the loop space case, which is treated in section 3. In section 4, we discuss fundamental F-theory and T-duality. In section 5, we review all the evidence we got so far for elliptic cohomology field strength in type II string theory, which comes from at least four different directions. In section 6, we define the version of elliptic cohomology-based partition function on type IIA which we need here. In section 7, we compare it to the partition function of F-theory defined in section 3 in the special case $\lambda = 0$. In the Appendix 8, we also discuss the $\lambda = 0$ restriction and show that despite its rather restrictive nature, it allows non-trivial examples.

## 2 A recapitulation of Witten's formula

In this section, we shall recall Witten's formula [24, 2] for the topological (Chern-Simons) term in the action of $M$-theory, along with the specific numbers involved. This will be useful in the next section, where numbers will be replaced by modular forms. Let $Y$ be an 11-dimensional spin manifold with a distinguished 4-dimensional class $G$. Then the vanishing

$$MSpin_{11}K(Z, 4) = 0$$

implies that there exists a 12-dimensional spin manifold $Z$ whose boundary is $Y$. Witten's formula for the Chern-Simons term in M-theory is

$$L_{CS} = \frac{1}{6}\int_Z G(G^2 - \frac{1}{8}(p_2(Z) - \frac{1}{4}p_1(Z)^2)). \tag{1}$$

Consider, then, a 12-dimensional spin manifold $Z$. Let $G \in H^4(Z, \mathbb{Z})$, and let $V$ be the adjoint complex $E_8$-bundle associated with $G$. Let $I_V$ be the index of the Dirac operator on $Z$, twisted by $V$. Let, also, $I_{RS}$ be the Rarita-Schwinger index, which in this case is the index of the Dirac operator on $Z$ twisted by the complexified tangent bundle minus 4 copies of the complex trivial (1-dimensional bundle). In this section, we recall the derivation, by direct computation, of following formula [2, 24]:

$$\frac{1}{6}\int_Z G(G^2 - \frac{1}{8}(p_2(Z) - \frac{1}{4}p_1(Z)^2)) = \frac{I_V}{2} + \frac{I_{RS}}{4}. \tag{2}$$

The reason for doing so is that in order to proceed with an analogue on loop space, we need a good understanding of this formula. One should mention that the significance of the formula (2) is that it is basically the motivation for setting (1); naively, one would only the leading cubic term in $G$. The correction term, to match the index expression, was (at least in the context of M-theory) derived by Hořava and Witten [6, 7] by considering anomaly cancellation in heterotic M-theory. The other term is a correction term which is needed to relate the formula to (2). This is needed to show the independence $\mod \mathbb{Z}$ of (1) of the choice of the cobordism $Z$ (actually, more precisely, one only uses (1) as a phase factor in the quantum partition function, and to this end one would need to know that (2) is always an integer. This is not true, but one multiplies the phase by the Pfaffian of the Rarita-Schwinger operator, which makes the phase always 1; see [2]).

To recapitulate the derivation of (2), recall that the Atiyah-Singer index theorem implies that

$$I_V = \int_Z ch(V)\hat{A}(Z). \tag{3}$$

Thus, we are essentially done if we can calculate $ch(V)$. Let us recall how this is done. The essential point is that this must be a polynomial in $G$ with integral coefficients. To discover this polynomial, it is not important that $Z$ be a 12-manifold, and in fact it is sufficient to consider the case $Z = BSU(2)$, and to take a map

$$BSU(2) \to BE_8$$

induced by a homomorphism $\phi : SU(2) \to E_8$ which is a canonical inclusion induced by a coroot. The map $\phi$ then induces an isomorphism on $\pi_3$, hence $B\phi : BSU(2) \to BE_8$ induces isomorphism of $\pi_4$, and hence on $H^4(?, \mathbb{Z})$ by the Hurewicz theorem.

But then we can further pull back via the map $BS^1 \to BSU(2)$ which is the classifying map of the group homomorphism $S^1 \to SU(2)$ given by the maximal torus. Therefore, if we pull back to $S^1$, we have an inclusion of $S^1$ to $E_8$ as a direct summand of the maximal torus (the choice of direct summand doesn't affect the answer). We need to calculate the Chern character of this representation of $S^1$. This means we must calculate how $S^1$ acts on the Lie algebra $e_8$. This can be figured out as usual by taking a coroot $h$, and taking the sum

$$\sum_{\alpha} z^{\langle h,\alpha\rangle} + rank(G) \tag{4}$$

where here $G = E_8$ and the sum is taken over all roots. Taking the standard presentation of the $E(8)$ lattice as generated by integral vectors $(a_1, ..., a_8)$ with $\sum a_i$ even (the $D(8)$ lattice) and $(1/2, ..., 1/2)$, we may take the (co)root $h$ to be $(1, 1, 0, ..., 0)$, so there is one root whose inner product with $h$ is 2, $2 \cdot 2 \cdot 6$ roots in the $D(8)$ lattice and $2^5$ roots with fractional coordinates (a total of 56 roots) whose inner product with $h$ is 1, an equal number of roots with opposite signs, and all remaining 126 roots having 0 inner product with $h$. Thus, (4) is

$$z^2 + z^{-2} + 56z + 56z^{-1} + 134, \tag{5}$$

and the corresponding Chern character is calculated by plugging in $z = e^{G/2}$, i.e. we get

$$e^G + e^{-G} + 56e^{G/2} + 56e^{-G/2} + 134. \tag{6}$$

Expanding, (6) is equal to

$$248 + 60G + 6G^2 + \frac{1}{3}G^3 + HOT. \tag{7}$$

Now regarding the $\hat{A}$-class, one has

$$\begin{array}{l}\hat{A}(Z) = 1 - \frac{1}{24}p_1(Z)+ \\ \frac{1}{5760}(-4p_2(Z) + 7p_1(Z)^2)- \\ \frac{1}{967680}(16p_3(Z) - 44p_1(Z)p_2(Z) + 31p_1(Z)^3) + HOT.\end{array} \tag{8}$$

Multiplying with (6) and restricting attention to elements of dimension 12, we obtain

$$\frac{I_V}{2} = \int_Z (\frac{1}{6}G^3 - \frac{1}{48}(p_2 - p_1^2/4)G - \frac{31}{15120}p_3 + \frac{13}{30240}p_1p_2 - \frac{1}{15120}p_1^3). \tag{9}$$

The Chern character of $TZ_{\mathbb{C}} - 4$ is

$$8 + p_1 + \frac{1}{12}(p_1^2 - 2p_2) + \frac{1}{360}(p_1^3 - 3p_1p_2 + 3p_3).$$

We get

$$\frac{I_{RS}}{4} = \int_Z \frac{31}{15120}p_3 - \frac{13}{30240}p_1p_2 + \frac{1}{15120}p_1^3, \tag{10}$$

so adding (9) and (10) gives (2).

# 3 Lifting the action from M-theory to F-theory

The above discussion suggests that to obtain a natural version of the action (2) which involves a modular parameter of an elliptic curve $\tau$, we should replace the $\hat{A}$-class in (3) by the Witten class with generating series

$$\frac{u/2}{sinh(u/2)} \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^n e^u)(1-q^n e^{-u})} = exp\left(\sum_{k>0} \frac{2}{(2k)!} S_{2k} u^{2k}\right). \tag{11}$$

Here we follow Zagier [25]. We put, as usual, $q = e^{2\pi i \tau}$, but to avoid confusion, here we replace the notation $G_k$ for Eisenstein series by $S_k$, so we have

$$S_\ell = S_\ell(\tau) = -\frac{B_\ell}{2\ell} + \sum_{n=1}^{\infty}\left(\sum_{d|n} d^{\ell-1}\right) q^n. \tag{12}$$

In discussing the geometric term of the F-theory effective action, we will first assume

$$p_1(Z) = 0. \tag{13}$$

This is a technical assumption, under which there exists the index on untwisted Dirac operator on loop space, see [22]. However, we shall remove the assumption later in this section, and derive the effective action in the general case, without any restriction on $p_1(Z)$. Now we would like an analogue of the action (2), but we realize immediately that there is a problem. What should one replace the $I_{RS}$ term by? Clearly, it is wrong to modify the Rarita-Schwinger index formula simply by replacing the $\hat{A}$ genus by the Witten genus. The reason is that the resulting formula would no longer compute an index on loop space (and consequently, for example, would not be modular). Instead, we should somehow involve the tangent bundle of the manifold $LZ$, but we do not know what the right analogue of the Rarita-Schwinger index is in that case, since $LZ$ is infinite-dimensional. Clearly,

some additional idea is needed to solve this problem; we return to this point briefly at the end of this section. For the purposes of the present paper, we choose a somewhat artificial (although, again, suggestive) solution. Recalling (10), we see that in the finite-dimensional case, the Rarita-Schwinger index serves to simply cancel the 12-dimensional term of the gravitational anomaly. In the loop case, we encounter a similar term. In the absence of knowing the exact form of the loop space index which would cancel that term, we simply take only the contribution of the $V$-bundle on $LZ$, and truncate the resulting loop index formula below the 12-dimensional term, assuming that we have a "Rarita-Schwinger"-type index on loop space which would cancel it. This is not so unreasonable for our purposes, since we are attempting to carry out an analysis analogous to [2], where the main focus is also on the $E_8$-bundle contribution. Then, we are only interested in the characteristic class associated with (2) up to $p_2$, so we get

$$1 - \frac{1}{6} S_4 p_2. \tag{14}$$

Thus, the modification of the action (2) in the presence of the restriction (13) is

$$\int_Z G(\frac{1}{6} G^2 - 5 S_4 p_2). \tag{15}$$

(When $q = 0$, $S_4 = -B_4/8 = 1/240$, so (15) reduces to (1).) Now let us explain the physical motivation for choosing this formula. The main idea is that, denoting by $W$ the Witten class, then

$$\int_Z ch(V) W(Z) \tag{16}$$

is supposed to be equal to the index of the Dirac operator on the loop space $LZ$ (consisting of smooth maps from $S^1$ to $Z$), twisted by the bundle $V$, pulled back to $LZ$. At present, this cannot be made mathematically rigorous, since one cannot rigorously define the index on loop space, but Segal [18] explains that one may conjecture an analog of the localization formula for equivariant index applies (Proposition 3.8 of [18]), and that this formula gives (16). In fact, this presents another small problem, since the obvious projection $LZ \to Z$ is by evaluation at a point, which is not equivariant, so one must argue why the pullback of the bundle $V$ to $LZ$ can be taken to be an $S^1$-equivariant bundle. However, one way to treat this is to note that the index of the Dirac operator is actually determined by localizing near fixed loops, where the problem does not arise.

Let us now turn to the question as to what the action should look like in the absence of the restriction (13). It is a standard procedure to try to cancel an anomaly by adding a gauge term. In the case of the Witten genus, as described in [22], the anomaly of the Dirac operator on $LZ$ can be cancelled by subtracting the bundle of loops on a "gauge bundle" $V$ on $Z$ which satisfies the condition

$$p_1(V) = p_1(Z). \tag{17}$$

However, which bundle to choose? A spin bundle is, of course, not characterized by its first Pontrjagin class. Additionally, the obvious candidate suggested by the term "gauge bundle", namely the adjoint vector bundle of an $E_8$-bundle associated with a 4-dimensional integral cohomology class, turns out to give the wrong answer (as can be checked using the Pontrjagin class calculation reviewed in the last section). It is worth commenting here that the term we are seeking is in fact not the usual string gauge term. What we are trying to do is cancel the anomaly associated specifically with lifting the action to the loop space, but the whole term we are studying here corresponds to the gauge term in M-theory action. In other words, we are trying to cancel an anomaly which, in this lift, arises within the M-theory gauge term itself.

Nevertheless, we can actually deduce what Pontrjagin classes the bundle $V$ must have. The general form of the F-theory action compactified to IIA spacetime must be

$$\int_Z G(\frac{1}{6}G^2 - 5S_4(p_2 - \frac{1}{4}p_1^2)). \tag{18}$$

To see this, disregarding the dimension of $G$, terms of topological dimension $2n$ must be coupled to modular forms of weight $n$; therefore, terms of topological dimension 4 are excluded, as there are no modular forms of weight 2. Alternately, any such term is a scalar multiple of $p_1$, which is excluded by the requirement of anomaly cancellation of the Dirac operator on loop space; the 8-dimensional term is identified by taking the $q \to 0$ limit, and is unique, as again there are no other modular forms of weight 4. The 12-dimensional term is excluded also by the $q \to 0$ limit, as there is no cusp form (modular form vanishing at $q = 0$) of weight 6. This last argument also confirms our Ansatz that the loop Rarita-Schwinger operator, whatever it is, must cancel the 12-dimensional term.

In other words, the action (18) is fixed. If indeed we were looking for a

gauge bundle $V$ accomplishing this transformation, it would have to have

$$p_1(V) = p_1(Z),\ p_2(V) = \frac{1}{4}p_1(Z)^2.$$

We do not know if if such a bundle indeed exists in general, although rationally (when $p_1(Z)$ is divisible by a certain integer), this is always possible by general homotopy-theoretical arguments, as $BSpin$ is rationally the product

$$\prod_{n>0} K(4n, \mathbb{Q}).$$

Why the index on loop space be related to F theory can be justified in the following way: thinking of $G$ as a field strength, the formula (2) can be thought of as a part of the action describing the dynamics of a 3-dimensional world volume in $Z$. Now considering the motion of the same world volume $A$ in $LZ$ is the same thing as considering the motion of $S^1 \times A$ in $Z$. This means that if we believe the index formula on loop space, then (15) describes the dynamics of a 5-dimensional field strength $G_5$ in $Z$ (more exactly its sector where the "time coordinate" of the world volume together with the additional $S^1$-coordinate form an elliptic curve of modulus $\tau$, similarly as when calculating the partition function of a fundamental string). This is a field one expects to see in F-theory. Denoting the corresponding potential by $A_4$, this was noted in [15], following [3]. It was further remarked in [15] that the reason the $G_5$-field appears is that the F-theory action provides a framework for unification of IIA and IIB string theories. Similarly as in [22], the geometric presence of the elliptic curve of modulus $\tau$ in the dynamics of $G_5$ is the physical cause of the modular invariance of the formula (15).

There is, however, a surprise contained in this analysis. In this picture, we do not see F theory spacetime as fibered over M theory. Rather, the connection is that if Y is a spin manifold interpreted as M theory spacetime, then the phase of its M-theory action (the Chern-Simons term) is computed by integrating (2) over a manifold $Z$ which is a spin cobordism between $Y$ and $\emptyset$. The modular formula (15) describes the action of a field which is "$S^1$-fibered over $G_4$" in the above sense, but which is present on $Z$. Therefore, we are not seeing a 12-dimensional manifold $Y \times S^1$, and carrying this further, we certainly are not seeing a 13-dimensional manifold whose boundary would be a 12-dimensional manifold in the present picture.

One more comment is in order about the formula (15). Recall again that we didn't discuss the Rarita-Schwinger index. Indeed, it is not obvious

what kind of analogue of the Rarita-Schwinger operator one should consider on loop space. Observing (9) and (10), however, suggests that we could solve this problem here simply by dropping the 12-dimensional term (not counting the dimension of $G$). Perhaps this points, however, to the deeper point that boundary phenomena become more complicated on loop space, since the "boundary" of the space of loops on a manifold cannot be identified simply with the loop space on the boundary. Even more fundamentally, the ultimate problem may be that we do not understand well gravity in F-theory. Although a possible supergravity was proposed for physical F-theory in [12], it is fair to say that our understanding of this situation is indeed in its infancy.

## 4 On fundamental F-theory and duality

In this section, let us look briefly at what our findings signify for the development of F-theory as a fundamental physical theory. In [3], Ferrara, Minasian and Sagnotti investigated the $F$-theory geometric coupling term

$$\int_{Z^{12}} A_4 \wedge G_4 \wedge G_4 \tag{19}$$

where $A_4$ is the 4-potential of type IIB lifted to 12 dimensions, and $G_4$ is the 4-form of 11-dimensional supergravity lifted to 12 dimensions (i.e. the field we considered above). Without going into details of their notation, let us note that they investigate subharmonic expansions of $A_3$, $A_4$ on $Z^{12} = M^6 \times CY$ of particular form, and show that (19) then gives an expected 6-dimensional coupling term, thus justifying that (19) must be present. They also note, by similar arguments, that correction terms of the forms

$$\int_{Z^{12}} A_4 \wedge I_8, \tag{20}$$

$$\int_{Z^{12}} G_4 \wedge I_8, \tag{21}$$

$$\int_{Z^{12}} G_4 \wedge G_4 \wedge G_4 \tag{22}$$

must be present with suitable coefficients.

Now one puzzling part of this picture is the relationship between the fields $G_4$ and $A_4$: [3] note that the fields $A_4$, $G_4$ are not independent in

their example, yet they do not derive an equation relating them. They propose that such equation should be supplied by the dynamics of F-theory. Yet, no such equation has emerged since.

One of the points of the present paper is to add evidence to the proposal that there exists a *fundamental* physical F-theory. This proposal was made in [12], as a natural conclusion following the investigation done in [13, 14, 15]. The idea is that theories in 12 dimension should be somehow unified to the same theory. In the present paper, we are investigating the manifold $Z^{12}$ which is a cobordism of a spacetime of M-theory. This is, of course, not the same thing as the elliptic fibration on IIB space considered in [3]. Yet, we propose that they are sectors of the same theory (later in this section, we shall make that more explicit).

For now, let us note that the field content we observed in $Z^{12}$ is analogous to the field content of [3]: we obtained $G_5$ as $G_4$ moving on the loop space of $Z^{12}$. We can visualise this as a 3-dimensional world volume $M_3$ in $LZ^{12}$. In terms of field strengths, this suggests that $G_4$, $G_5$ are components of a unified field

$$G \in H^4(LZ^{12}, \mathbb{Z}). \tag{23}$$

(In fact, if we really wish to formulate a fundamental theory, we must note that $E_8$-bundles on $LZ^{12}$ are no longer classified by $H^4$, so we must also require the additional quantization condition that (23) lifts to an $E_8$-bundle.)

Now in our setting, the Chern-Simons term of M-theory seems to have a natural generalization to fields of the form (23), which can be considered as a modification of (19). The topology of this situation can be interpreted as a tie between $G_4$ and $A_4$ ($A_4$ being a potential corresponding to the field strength $G_5$), but not as strong as a coupling equation. We therefore conclude that the relationship [3] need is an attribute of examining a particular sector of the theory, just as the restriction on $G_4$ assumed in [2] (that it come from a cohomology class of $X^{10}$).

Of course, one important point we must address is relating our setting to the setting of [3]. There are important differences, for example the fact that we consider $Z^{12}$ to be a spin cobordism of M-theory spacetime $Y^{11}$, i.e. a manifold with boundary. However, there is a path we can use to relate our theory to an elliptic curve fibration of IIB, namely a generalization of T-duality relating IIA and IIB: when IIA is considered on a space of the form $X^9 \times S^1$, by shrinking the $S^1$ to a point, that coordinate disappears, but a string wrapped around the $S^1$ becomes light, signalling the opening

of another dimension, thus giving the T-dual IIB theory on $X^9 \times S^1$.

Now it is impossible to apply such T-duality naively to M-theory, because M-theory lacks fundamental strings; it has however 2-branes, and one compactifies M-theory on $S^1 \times S^1$ and shrinks this 2-torus to a point, the 2-brane wrapped on the torus becomes light and new dimension opens up, giving 10-dimensional IIB-theory.

In our settings, the situation is additionally complicated by the fact that we are considering the $K$-theory field strength corrections to type II strings, so we must take into account spin structure. Consider type IIA string theory on $X^9 \times S^1_{NS}$, which is M-theory on

$$X^9 \times S^1_{NS} \times S^1_R. \tag{24}$$

Now we know, however, that this is really F-theory on a spin-cobordism $Z^{12}$ between (24) and 0. In the special case (24), however, a special kind of $Z^{12}$ can be proposed, namely

$$X^9 \times E' \times S^1_R \tag{25}$$

where $E'$ is a spin-cobordism from $S^1_{NS}$ to 0. Then shrinking the size of $S^1_{NS}$ to 0 in the boundary (while preserving the bulk) corresponds to gluing a disk to $E'$. Denoting the corresponding closed surface by $E$ (which can be an arbitrary Riemann surface, in particular an elliptic curve), we get the corresponding bulk F-theory on

$$X^9 \times E \times S^1_R. \tag{26}$$

Thus, assuming the T-duality mechanism extends to this cobordism context, we exhibited a "T-duality" between the $F$-theory with boundary $M$-theory and $F$-theory fibered on IIB spacetime in this case.

Let us also note however that there should be another "self-T-duality" of the fibered $F$-theory. Consider in particular the F-theory on

$$X^9 \times \prod_{i=1}^{3} S^1_R \tag{27}$$

which is a special case of (26). Then from the evidence we saw, this theory should have a 2-brane $M_2$ and a 3-brane $M_3$ where the relationship (23) becomes

$$M_3 = M_2 \times S^1.$$

In particular, then, $M_2$ can be wrapped on

$$\prod_{i=2}^{3} S^1_R$$

and $M_3$ on

$$\prod_{i=1}^{3} S^1_R.$$

If we shrink the radius of the first copy of $S^1_R$ to 0, then, $M_3$ will lose a dimension, but $M_2$ will expand by the new dimension, and we see than that the system $(M_2, M_3)$ is self-dual.

## 5 The evidence for elliptic cohomology field strength in string theory

Very briefly, much of our present nonperturbative knowledge of string theory came from the study of $D$-branes. Polchinski [16] noticed that the RR-fields in free type II supergravity can be interpreted as $p$- dimensional differential forms ($p$ odd for type IIA and even for type IIB). He proposed that in non-perturbative theory, these forms correspond to submanifolds which are $D$-branes, i.e. boundary sets of open strings. The $D$-brane is however considered as a dynamic object with 1 timelike dimension, so a $p$-form corresponds to a $p-1$-brane. On the other hand, the forms are potentials, so a $p$-form corresponds to $p+1$-dimensional field strength denoted by $F_{p+1}$ or $G_{p+1}$. The relationship between potential and field strength has not been completely mathematically clarified in general dimension, although various models have been proposed which generalize both the exterior differential and the connection-curvature relationship. There are other complications to the story, such as the distinction between electric and magnetic charges in relationship to Poincare duality, but we needn't discuss these here. We refer the reader to [17] for a survey.

Witten however noticed that this story needed a correction (see [23]). $D$-branes carry vector bundles of Chan-Paton charges, and in fact $D$-brane processes correspond to addition of vector bundles, so in other words the charges of stable $D$-branes should be elements of $K$-theory. This should then be true also of the field strengths $G_k$. It leads to the conclusion that

all field strength sources in type II string theory which have apparently different dimensions can be added and expressed in terms of one $K$-theory class $G \in K^0(X)$ for $X$ of type IIA and $G \in K^1(X)$ for $X$ of type IIB. A free field approximation partition function for type IIA and IIB based on this is calculated in [2], and takes the form of a theta function. The main result of [2] is showing that this matches the partition function of M-theory (with all the subtlety contained in the discussion of phase).

The first evidence that an additional correction is needed was in fact given in the same paper, [2]. There, Diaconescu, Moore and Witten noticed that in the presence of the $H_3$-field in type II string theory, the correct analog of the $K$-theory charge lies in twisted $K$-theory where the twisting is given by $H_3$. But this violates S-duality in type IIB: the first k-invariant, or "Atiyah-Hirzebruch spectral sequence differential" in twisted $K$-theory is not invariant with respect to the modular transformations between $H_3$ and $F_3$ which $S$-duality predicts. In fact, in [14], the the first author and Sati showed that there exists no modified definition of twisted $K$-theory which would remedy this problem without introducing other homotopy groups (i.e. other fields).

Why is S-duality violated in the $K$-theory field strength? An explanation was offered in [15]: the match between the IIA and M-theory partition functions in [2] can be interpreted as confirmation of Witten's conjecture that IIA string theory can be interpreted as compactification of 11-dimensional M-theory on a circle. This is a refinement of his earlier observation that IIA theory at strong coupling is M-theory. Considering the total field strength in $K$-theory involves bringing out this non-perturbative view of the theory. But now considering the same field strength in type IIB suggests (as can be further substantiated by T-duality) the same construction, i.e. viewing IIB theory as an 11-dimensional "T-dual" of M-theory compactified on a circle. From that point of view, however, it is apparent why such model should violate S-duality, since such duality should partner the extra dimension with another: this is the proposal of Ferrara, Minasian and Sagnotti [3] which suggests to consider IIB instead as a compactification of 12-dimensional F-theory on an elliptic curve whose dimensions are permuted by S-duality.

Returning however back to type IIB, what kind of refinement of the $K$-theory field strength could possibly capture the additional 12'th dimension, and match the partition function of F-theory? In [14], a natural candidate emerged in the form of elliptic cohomology. Elliptic cohomology (more precisely the Hopkins-Miller spectrum $TMF$) was suggested from at least four

different directions. First of all, its torsion free classes are modular forms, so this suggests modularity. However, we must be careful not to identify this modularity with S-duality in the most obvious way: as we saw above, the modularity of $TMF$ is a phenomenon in two of the four dimensions of a world volume, while $S$-duality is a spacetime phenomenon. If we propose a IIB field strength based in $TMF$, then its topological modularity will be present even in the analogous field strength of IIB, which must in fact exist by T-duality.

More strongly than the modularity, $TMF$ has the property that turning on the $H_3$-field does not cause twisting, but merely multiplication by a certain $TMF$ class (see [14]). This is encouraging, because the twisting itself is a puzzle. If we want to turn on both the $H_3$ and $F_3$ field in type IIB, we want the field strength to be group-valued. But this is not the case for twisted $K$-theory considered simultaneously at different twistings: It seems to require a single generalized cohomology theory which is not twisted by turning on these fields.

The next piece of evidence for elliptic cohomology came in [9], [10] where Hu and the first author attempted a more rigorous mathematical formalism for $D$-branes. In that paper, it was showed that while a simpler model of $D$-branes exists with Chan-Paton charges in vector spaces, the full model which captures all possible anomalies of $D$-branes has charges in 2-vector spaces, the theory of which is linked with elliptic cohomology.

Finally, the first author and Sati [13, 14, 15] noticed that an elliptic cohomology-valued field strength and partition function can in fact be defined in type IIA and IIB in analogy with the $K$-theory picture. For technical reasons, actually, these papers work with real-oriented elliptic cohomology, which has only level 3 modularity (i.e. its torsion free classes are forms defined on the moduli space of elliptic curves with level 3 structure. Below we propose a correction in the case of type IIA based on the theory $TMF$, which has full modularity. This partition function matches (to the extent considered) the F-theory partition function.

One should find out if indeed the $TMF$-partition function in IIB matches the F-theory partition function, and therefore preserves S-duality. This problem will be considered in future work. In the present paper, we specialize to IIA, where the discussion is easier because of the huge amount of work done in [2].

We should also mention that the original motivation for the elliptic partition function in [13] was that the $W_7(X)$ anomaly of type IIA string theory on $X$ turns out to be precisely the obstruction to orientability of spacetime $X$ with respect to complex-oriented elliptic cohomology. However, one needs orientability of $X$ with respect to real-oriented elliptic cohomology to get a quadratic structure of the kind needed for defining a theta function. The obstruction to such orientability is $w_4$. Orientability of $X$ with respect to $TMF$, which we consider here, has the obstruction $\lambda \mod 24$. These are successively stronger conditions. The 4-dimensional obstructions were previously known to arise in type I and heterotic string theories, but not in type II. Therefore, this suggests that the F-theory interpretation should unify all types of string theories. In the present paper, we operate with index on loop space, which requires the vanishing of $\lambda$ outright (the condition (13)). This is why we impose this strongest condition throughout the present paper.

## 6 The $TMF$ partition function in type IIA.

In [13, 14, 15], Kriz and Sati proposed analogues of partition functions of type IIA (and IIB) string theory based on elliptic cohomology. A part of the reason however a definitive version could not be proposed was that it was not clear which elliptic cohomology one should use. We remarked that likely the Hopkins-Miller theory $TMF$ (which stands for topological modular forms) should be used, but couldn't do that in part because $TMF$ is not an elliptic cohomology theory in the ordinary sense (rather, it is a "homotopy inverse limit of all elliptic cohomology theories", while a universal elliptic cohomology theory does not exist).

In the present paper, however, we have proposed a much more concrete formula for the relevant sector of F-theory action, and also understand its physical interpretation better. In the present context, we have seen that indeed the index of Dirac operator on loop space occurs, which by Witten's conjecture is related to the $TMF$ characteristic class. Therefore, we conclude that we must indeed use $TMF$ itself.

Now we are mostly interested in 2-torsion considerations, for which purpose we can replace $TMF$ by its 2-complete version in the sense of homotopy theory. By results of Hopkins and Mahowald [5], this spectrum (which they call $EO_2$) can be obtained as a certain fixed point spectrum $(E_2)^G$ where $E_2$ is Landweber elliptic cohomology with coefficients $W_2[[u_1]][u, u^{-1}]$, $W_2$ are the Witt vectors, $dim(u) = 2$, and $G = SL_2(\mathbb{F}_3)$ acts on this spectrum

in a suitable way. In [5], the most explicit computation of the 2-completed homotopy groups of $TMF$ are actually given in Theorem 9.11, which describes its connective version. One obtains $TMF$ from this description by inverting $v_2^{32}$.

In order to define a $TMF$-valued $\theta$-function, we need a "complex" version $TMF_{\mathbb{C}}$ of $TMF$ in the same sense as complex $K$-theory is a complex version of real $KO$-theory. It is not completely obvious how to do this for $TMF$ for the following reason: one has a standard $\mathbb{Z}/2$-action on $E_2$ which corresponds to taking inverse in the formal group law, and in [8] we called the corresponding $\mathbb{Z}/2$-equivariant spectrum $E\mathbb{R}_2$. (It is then natural to call the fixed point spectrum $EO_2$, which unfortunately conflicts with the notation of [5].) Now the problem is that the inclusion $\mathbb{Z}/2 \subset G$ is simply the center (which of course doesn't split off), so it is not obvious how, simply by using the equivariant structure, we could "forget" the action of this $\mathbb{Z}/2$-group while remembering the rest of the $G$-action.

Fortunately, there is a solution stemming from homotopy theory. Let us denote by $\eta$ the generator of the stable homotopy group $\pi_1(S) = \mathbb{Z}/2$, and denote (in agreement with [5]), by $M(\eta)$ the spectrum which is the cofiber of this map. Now in $K$-theory, we have the relation

$$KO \wedge M(\eta) \simeq K. \tag{28}$$

Now when we invert the class $(v_1)^4$ in the ring spectrum $TMF$, forming a spectrum

$$v_1^{-1}TMF, \tag{29}$$

we obtain a direct (wedge) sum of copies of orthogonal $K$-theory $KO$. Our idea is that the spectrum (29) will be enough to recover our partition function: we are using index-theoretic tools, which means that we are essentially using $K$-theory anyway. Therefore, it is natural to simply put

$$v_1^{-1}TMF_{\mathbb{C}} = v_1^{-1}TMF \wedge M(\eta). \tag{30}$$

We know that (30) is a commutative associative ring spectrum (giving a cohomology theory which, on spaces, has graded-commutative associative multiplication), since the same is already true for the spectrum $M(\eta)$. Now the spectrum $v_1^{-1}TMF_{\mathbb{C}}$ is a direct (wedge) sum of copies of complex $K$-theory $K$ (to see this, one can compute, in a stadard way, using e.g. the Adams-Novikov spectral sequence, the homotopy of the function spectrum

$F(v_1^{-1}TMF_{\mathbb{C}}, K)$. Another observation is that $v_1^{-1}TMF_{\mathbb{C}}$ is a complex oriented spectrum. Furthermore, for

$$u \in v_1^{-1}TMF^0_{\mathbb{C}}(X), \tag{31}$$

we have a class

$$u\overline{u} \in v_1^{-1}TMF^0(X). \tag{32}$$

Looking at Theorem 9.11 of [5], we see that in the homotopy of $TMF$, there is copy of the connective $k$-theory homotopy shifted by the generator $v_1^4$ which is the Bott class in dimension 8 (tensored with $\mathbb{Z}[v_2^4, v_2^{-4}]$, but we don't use that at the moment). The point is however that there is the class

$$v_1^4\eta^2 \in TMF_{10} \tag{33}$$

analogously as in $KO(10)$. Moreover, in $TMF_{\mathbb{C}}$, the

$$v_1^4 kO_*[v_2^4, v_2^{-4}]$$

summand is replaced by

$$v_1^4 k_*[v_2^4, v_2^{-4}],$$

so there also is the non-torsion class

$$v_1^5 \in (TMF_{\mathbb{C}})_{10}. \tag{34}$$

But this suits us well. Now we can take advantage of the fact that, just as in [13], the $\mod 2$ index can be interpreted as cap product with an orientation class

$$[X] \in TMF_{10}(X) \tag{35}$$

where $X$ is 10-dimensional spacetime (which is assumed to be a compact manifold which is $TMF$-orientable by (13), [1]).

In the present paper, we only consider a contribution to the modular partition function of type IIA string theory based on a 4-dimensional cohomology class

$$G \in H^4(X^{10}, \mathbb{Z}), \tag{36}$$

analogously to Chapter 7 of [2]. Additionally, we agreed to invert $(v_1)^4$, which should preserve all information which we can recover by index theory. Now [2] start with a lift of $G$ to $K^0(X)$ via the Atiyah-Hirzebruch spectral sequence. We need to form the lattice $L$ spanned by all such classes $G$ and their duals $*G$. Now we know from the above observations that the classes

in $L$ in fact automatically lift to $v_1^{-1}TMF_{\mathbb{C}}$. Now we can consider the "phase factor" for

$$u \in v_1^{-1}TMF_{\mathbb{C}}$$

given by

$$j(u) = \langle u\overline{u}, [X]_{v_1^{-1}TMF}\rangle \in v_1^{-1}TMF_{10} \tag{37}$$

(see (32)). Since we remarked that $v_1^{-1}TMF$ is a wedge sum of copies of $KO$-theory, we can also interpret this as a mod 2 index where, at least morally, again $\hat{A}$ is replaced by the Witten class $W$. Conjecturally, this should also equal the mod 2 index on the loop space of $X$.

There is a subtle problem with this definition. Coefficients of $v_1^{-1}TMF$ are not exacly modular forms. They are modular functions, but inverting $v_1$ corresponds to inverting $S_4$, i.e. allowing a pole at $\tau = \sqrt[3]{-1}$ (which is a zero of $S_4$). If we want to show that we have an actual modular form without singularities, we need to know that on the right hand side of (37), we actually get a $v_1$-invertible class in $TMF_{10}$ itself, without a $v_1$ denominator (i.e. a $v_1$-integral class). But to this end, we use the definition in [13]: $u$ also lifts into the complex-oriented generalized cohomology theory $E_2$. This is because $E_2$ is a complex-oriented cohomology theory. But a 4-dimensional integral class on a 10-manifold cannot have a differential $d_{>3}$ in the Atiyah-Hirzebruch spectral sequence ($d_3$ is the obstruction to lifting to $k$-theory). The reason $d_4$ is impossible is that it would land in $H^8(X,\mathbb{Z})$ which has a Poincare coupling with $H^2(X,\mathbb{Z})$. But classes in $H^2(X,\mathbb{Z})$ are represented by maps to $\mathbb{C}P^\infty$, and the fundamental class in $H^2(\mathbb{C}P^\infty,\mathbb{Z})$ lifts to any complex-oriented cohomology theory, which makes it impossible for the dual class in $H^8(X,\mathbb{Z})$ to be killed by an AHSS differential. The argument for $d_{>4} = 0$ is analogous.

Now in [13], we constructed an analogue of the function (37) which lies in $(E\mathbb{R}_2)_{10}$ where $E\mathbb{R}_2$ is the real version of the elliptic cohomology spectrum $E_2$. Now the coefficients of $E\mathbb{R}_2$ are not modular, but the following argument can be made: The coefficients of $v_1^{-1}TMF$ map injectively to the coefficients of $v_1^{-1}E\mathbb{R}_2$. Moreover, the intersection of the image with the image of the coefficients of $E\mathbb{R}_2$ is equal to the image of the coefficients of $TMF$. To see this, the statement is certainly true rationally, i.e. when we tensor with $\mathbb{Q}$. To obtain the integral statement, we argue that the torsion free and $v_1$-torsion free factor $\Phi_1$ of $TMF_*$ is a direct summand of the torsion free and $v_1$-torsion free summand $\Phi_2$ of $(E\mathbb{R}(2))_*^{\mathbb{Z}/2}$. Indeed, the latter graded

abelian group is

$$(\mathbb{Z}\{1, 2v_1^1\}[v_1^4, v_2^2] \oplus v_2^2\mathbb{Z}\{1, v_2^2\}[v_2^4])[v_2^{-4}], \tag{38}$$

while the former is

$$(\mathbb{Z}[v_2^{32}] \oplus v_1^4\mathbb{Z}\{1, 2v_1^2\}[v_2^4])[v_2^{-32}]. \tag{39}$$

We see that (39) is a direct summand of (38). But this means that

$$\Phi_1 = \Phi_2 \cap (\Phi_1 \otimes \mathbb{Q}).$$

The right hand side is, by our rational observation, the intersection of $\Phi_2$ with $v_1^{-1}TMF \otimes \mathbb{Q}$, which contains $\Phi_2 \cap v_1^{-1}TMF$, which is therefore contained in $\Phi_1$. The converse inclusion is trivial.

Therefore, integrality with respect to $v_1$ of the class (37) can be tested by looking at $E\mathbb{R}_2$, which was done in [13].

# 7 Comparing F-theory and IIA.

First, let us see what we have computed. Starting with the lattice $L$, we can take the pairing whose imaginary part, similarly as in [2], Section 7.1, is given by

$$\omega(x, y) = \langle x\overline{y}, [X]_{v_1^{-1}TMF_{\mathbb{C}}}\rangle \tag{40}$$

where on the right hand side, we are really considering the lifts of $x, \overline{y}$ to $v_1^{-1}TMF_{\mathbb{C}}$. We observe, however, that the right hand side is given by the formula

$$\int_X ch(x\overline{y})W(X). \tag{41}$$

Now since we are assuming (13), the lowest dimension in which $W(X)$ differs from $\hat{A}(X)$ is dimension 8, while the bundles $x$, $\overline{y}$ which we choose to represent the cohomology classes, have $c_1 = 0$ (see [2], Section 7.2). Thus, we see that we may in effect replace $W(X)$ by $\hat{A}(X)$ in (41), and the only new information our partition function will gain will be in the $\mod 2$ index, i.e. in calculating $j(x)$.

Now in the last section, we have defined $j(x)$ as an element of the image of $TMF_{10}$ in $v_1^{-1}TMF$. This means that the answer is a 'modular form' (with possible pole at $q = 0$), but it is also 2-torsion! At first, this seems like a puzzle both mathematically and physically. Mathematically, clearly one

cannot state modularity the usual way in terms of the modular parameter $\tau$. Yet, the very existence of $TMF$ and its "modularity" shows that some interpretation must exist. The interpretation, of course, is that the class constructed is of the form $\eta^2\phi$ where $\eta$ is the generator of $\pi_1(S)$ and $\phi$ is a modular form with integral coefficients: one can consider elliptic curves over $\mathbb{Z}$, and then one has well defined modular forms with inverted discriminant (modular functions), which, as recalled for example in [5], form the ring

$$R[\Delta^{-1}] \tag{42}$$

where $R$ is the ring of modular forms, given by

$$R = \mathbb{Z}[S_4, S_6, \Delta]/(1728\Delta = S_4^3 - S_6^2). \tag{43}$$

Of course, we know that not all of the modular forms (42), even after multiplication by $\eta^2$, are topological modular forms. Therefore, the way we must interpret $j(x)$ in the presence of $TMF$ is that after plugging integral values into topological modular forms, or, more precisely, choosing a homomorphism

$$h : TMF_* \to \mathbb{Z}/2, \tag{44}$$

$j(x) = j_h(x)$ becomes a well defined integer $\mod 2$. We then obtain a partition function with phase

$$\phi^h_{IIA} \tag{45}$$

in the theta function

$$exp(-i\pi Re\tau(\theta_h/2)) \sum_{x\in\Gamma_1} exp(i\pi\tau(x+\theta/2))\Omega_h(x) \tag{46}$$

indexed by the homomorphisms $h$. Here the formula (46) is precisely analogous to formula (7.11) of [2], where we pointed out with the subscript $h$ the terms which depend on that homomorphism. Physically, we must conclude that also, a choice of the discrete homomorphism (44) is needed to make the phase (45) well defined. We must therefore conclude that from the IIA point of view, only a discrete set of choices of modular parameter are allowed, and that restrictions on the geometry of the elliptic curve allowed in thus "quantized". The explanation is that the F-theory action term gotten as a loop version of the Chern-Simons term is geometrical, but in general not topological: there is therefore no reason why one should be able to recover the corresponding partition function for IIA. For special values of the

modular parameter, however, there is more symmetry and we shall see that a relation does hold.

Let us now set out to carry out an analog of the comparison [2] between type IIA and M-theory phases for modular type IIA and F-theory phases. Unfortunately, we only know how to do the easiest case, namely when the condition (13) holds. A brief discussion of the significance of this restriction is given in the Appendix. Under this restriction, much of the discussion is directly analogous to Section 7 of [2]. In fact, the discussion somewhat simplifies due to that fact that we assumed (13). In any case, we start with a class

$$G \in H^4(X, \mathbb{Z}) \tag{47}$$

which has a $K$-theory lift $x$. We represent, again, $x = E - F$ where $F$ is a trivial rank 5 bundle. In analogy with formula (7.28) of [2], we obtain the formula

$$\Omega_h(x) = (-1)^{q_L(ad(E))+I(E)} \tag{48}$$

where $q_L$ is the mod 2 loop index and $I(E)$ is ordinary index. Note that the use of ordinary index is justified by the above dimensional considerations. Similarly, the F-theoretical phase is, in analogy with formula (7.29) of [2],

$$(-1)^{f_h(a)} = (-1)^{q_L(ad(E))+I(\wedge^2 E)}. \tag{49}$$

Therefore, one can carry out a comparison of phases precisely analogous to [2], Section 7.7, noting that the formula of all index terms other than the mod 2 index are the same as there, and that the terms given by the mod 2 index are already matched by comparison of the formulas (48), (49). Therefore, we conclude that for these moduli $\tau$ for which the F-theory phase is only $\pm 1$, there is indeed a match between the phases of the $TMF$-based type IIA partition function and the F-theory phase.

## 8 Appendix: Discussion of the condition $\lambda = 0$

Let us discuss in this section the condition (13), when the ungauged Dirac operator on loop space is anomaly free. This is the special case in which we proved the Kriz-Sati conjecture in the last section. The restriction was not needed elsewhere in the paper.

First, let us comment that the condition is actually more accurately formulated as follows: in a spin manifold, there is a 4-dimensional characteristic

class $\lambda$ such that $2\lambda = p_1$. The more precise formulation of the condition (13) is

$$\lambda(Z) = 0. \tag{50}$$

In mathematics, in the context of elliptic cohomology, manifolds satisfying (50) became known as *string manifolds*, a term which since has been occasionally used in physics as well.

From the point of view of classical string theory, however, this is a serious misnomer, since the condition actually excludes many interesting vacua, for example complete intersection Calabi-Yau 3-folds: Let us, indeed, look at a complete intersection $K$ of $n-3$ hypersurfaces in $\mathbb{C}P^n$. If these are given by constraints $F_1, ..., F_{n-3}$ of degrees $\ell_1, ...\ell_{n-3}$, then the total Chern class of $K$ is

$$c(K) = (1+a)^{n+1}/(1+\ell_1 a)...(1+\ell_{n-3}a) \tag{51}$$

where $a$ is the generator of $H^2(\mathbb{C}P^n)$. The condition for Calabi-Yau is $c_1 = 0$, i.e.

$$n+1 = \sum_{i=1}^{n-3} \ell_i. \tag{52}$$

(We note here that in dimensions $\leq 6$, the restriction map $H^*(\mathbb{C}P^n, \mathbb{Z}) \rightarrow H^*(K, \mathbb{Z})$ is an injection: The fundamental homology class $[K] \in H_6(K)$ maps to a homology class Poincare dual to $c_1(F_1)...c_1(F_{n-3}) = a^{n-3}\ell_1...\ell_{n-3}$, which is non-zero, hence $a^3$ restricts to non-zero. Hence, so must $a, a^2$ by multiplicativity.) Now from (51), $c_2(K) = 0$ gives

$$\sum_{i<j} \ell_i \ell_j = \frac{n(n+1)}{2}. \tag{53}$$

So from (52), (53),

$$(n+1)^2 = \sum_{i=1}^{n-3} \ell_i^2 + \sum_{i<j} 2\ell_i\ell_j =$$

$$\sum_{i<j} \tfrac{\ell_i^2+\ell_j^2}{n-4} + \sum_{i<j} 2\ell_i\ell_j \geq$$

$$(\sum_{i<j} 2\ell_i\ell_j)\tfrac{n-3}{n-4} = n(n+1)\tfrac{n-3}{n-4}.$$

This gives $n^2 - 3n \leq n^2 - 3n - 4$, which is a contradiction.

This is, of course, only a most basic example of a fairly standard computation in this context. In fact, one may show more strongly that Calabi-Yau 3-folds in the strong sense, i.e. 6-dimensional manifolds with a metric which has $SU(3)$-holonomy, are flat (see [11, 21]).

On the other hand, it is not difficult to show by general homotopy-theoretical methods that 6-dimensional homotopy Calabi-Yau manifolds (also sometimes referred to as weakly Calabi-Yau, i.e. complex manifolds with $c_1 = 0$) satisfying (50) exist: Let us look at the cobordism ring $MU\langle 6\rangle$ of stably weakly complex manifolds with $c_1 = c_2 = 0$. Let us consider the situation rationally (more precise arguments are of course available, but would take longer). By the Thom isomorphism, one has

$$H^*(MU\langle 6\rangle, \mathbb{Q}) = H^*(MU\langle 6\rangle, \mathbb{Q}) = \mathbb{Q}[c_3, c_4, ...] \tag{54}$$

where $BU\langle 6\rangle$ is the classifying space of complex bundles satisfying $c_1 = c_2 = 0$. Now $n$-dimensional stably weakly complex manifolds (i.e. manifolds with complex structure on the stable normal bundle) satisfying $c_1 = c_2 = 0$ are classified by the $n$'th stable homotopy group

$$\pi_n MU\langle 6\rangle. \tag{55}$$

When tensoring with $\mathbb{Q}$, however, stable homotopy groups coincide with homology, so (54) shows that there are non-trivial elements, in fact elements $M$ which project non-trivially to $\pi_n(MU)$. That group is detected completely by Chern numbers.

Therefore, we certainly know that there are complex manifolds $K$ with $c_1(K) = c_2(K) = 0$ and which have non-zero Chern number $[c_3(K)]$, i.e. are not complex-cobordant to 0. While this shows that there are 6-dimensional Calabi-Yau spaces to which our proof applies, this is however of little interest since the statement trivializes in this case.

However, one can obtain non-trivial examples with 8-dimensional homotopy Calabi-Yau manifolds. In this case, because we assumed the vanishing of $c_1$, $c_2$, the only possibly non-trivial Chern number is $c_4[M]$ which shows that we must have

$$c_4(M) \neq 0 \in H^8(M). \tag{56}$$

Recall further that the total Pontrjagin class of a complex manifold has the form $c(M)\overline{c(M)}$ where $c(M) = 1 + \sum c_i(M)$ and $\overline{c(M)} = 1 + \sum(-1)^i c_i(M)$.

In view of vanishing of $c_1$, $c_2$, we see that in our case $p_2 = 2c_4$ and thus (56) implies $p_2 \neq 0$, so there are non-trivial 8-dimensional Calabi-Yau vacua to which our proof applies.

Although we proved only the easiest case of the Kriz-Sati conjecture here, nevertheless this discussion may indicate that we are ultimately touching on the question of physical vacua of F-theory. Although that subject is not well understood, it appears that other vacua than coming from 6-dimensional Calabi-Yau spaces are interesting in that case ([12]).

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